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# Tutorial 5 - Exhaustible Resources - Optimality SOLUTION

**Q1:** The social optimization problem is the following:

$$\begin{cases} \max_{\{s_t\}} & \int_0^\infty 2c_t^{1/2} e^{-\rho t} dt \\ s.t. & \dot{S}_t = -s_t \quad , \quad S_0 = \hat{S} \\ & s_t \geq 0 \end{cases}$$

Concentrating upon the interior case, where $s_t > 0$, the corresponding Hamiltonian is:

$$\mathcal{H}_t = 2(\bar{x} + s_t)^{\frac{1}{2}} e^{-\rho t} - \lambda_t s_t$$

Hence the first conditions read:

$$\frac{\partial \mathcal{H}_t}{\partial s_t} = 0 \implies c_t^{-\frac{1}{2}} e^{-\rho t} = \lambda_t$$
$$\frac{\partial \mathcal{H}_t}{\partial S_t} = -\dot{\lambda}_t \implies \dot{\lambda}_t = 0 \implies \lambda_t \equiv \lambda \ (cste)$$

These conditions imply together that the marginal utility of consumption must be constant in discounted terms along an optimal path. This is the Hotelling rule for optimal plans. The society must be indifferent at each time between consuming more or less resource today than tomorrow. This can be explained by an arbitrage argument showing that an increase of consumption over some first time interval gets an utility improvement which must be balanced by the utility decrease resulting from a symmetric decrease of consumption over some posterior time interval.

**Q2:** We get from the first order conditions the expression of $c_t$:

$$c_t = \lambda^{-2} e^{-2\rho t}$$

Using the definition of $c_t$, we conclude that:

$$s_t \equiv s(t, \lambda) = \lambda^{-2} e^{-2\rho t} - \bar{x}$$

It appears immediately that $s(t, \lambda)$ is a strictly decreasing function of $t$ and $\lambda$. We conclude that the extraction of the non renewable resource must be completed in finite time.

**Q3:** Since the optimal consumption path must be a continuous function of time, it results that $s_T = 0$. Using the previous expression of $s_t$ evaluated at $t = T$, we get:

$$0 = \lambda^{-2} e^{-2\rho T} - \bar{x}$$

which gives $T$ as a function of $\lambda$ :

$$T \equiv T(\lambda) = -\frac{1}{2\rho} \log(\bar{x}\lambda^2)$$

Note that $T \geq 0$ implies that $\lambda \leq \bar{\lambda} \equiv \bar{x}^{-\frac{1}{2}}$. $T(\lambda)$ is a strictly decreasing function of $\lambda$.

**Q4:** We get from the initial stock condition:

$$\int_0^T s_t dt = \hat{S}$$

Hence inserting the expression of $s_t$, we obtain:

$$\lambda^{-2}\{\frac{e^{-2\rho t}}{-2\rho}|_0^T\} - \bar{x}T = \hat{S}$$

that is:

$$\lambda^{-2}\{\frac{1-e^{-2\rho T}}{2\rho}\} - \bar{x}T = \hat{S}$$

**Q5:** Using the expression of $T$ we get:

$$\lambda^{-2} = \bar{x}e^{2\rho T}$$

Hence substituting into the stock condition, we obtain:

$$\bar{x}\{\frac{e^{2\rho T}-1}{2\rho}\} - \bar{x}T = \hat{S}$$

that is:

$$\frac{\hat{S}}{\bar{x}} = \frac{e^{2\rho T}-1}{2\rho} - T$$

Let:

$$f(x) = \frac{e^{ax}-1}{a} - x$$

We recognize the expression appearing in the right hand side of the stock relation between $\hat{S}$, $\bar{x}$ and $T$ where $a = 2\rho$. It is easily checked that :

$$f(0) = 0 \quad , \quad f'(x) = e^{ax} - 1 > 0 \quad f''(x) = ae^{ax} > 0$$

Thus, the function $f(.)$ being a monotonic strictly increasing function of $T$, the stock relation defines a unique value of $T$, function of $\hat{S}$ and $\bar{x}$ through the stock condition.

**Q6:** Total differentiation gets:

$$\frac{d\hat{S}}{\bar{x}} - \frac{\hat{S}}{\bar{x}^2}d\bar{x} = f'(T)dT$$

hence:

$$\frac{\partial T}{\partial \hat{S}} = \frac{1}{\bar{x}f'(T)} > 0 \quad , \quad \frac{\partial T}{\partial \bar{x}} = -\frac{\hat{S}}{\bar{x}^2 f'(T)} < 0$$

It appears that an increase of the non renewable resource stock results in a longer extraction length and that an increase in the available flow of the renewable resource results in a shorter exhaustion length. The first result is a consequence of concavity and discounting. Under discounting and concavity it does not pay for the society to increase the exploitation of the non renewable resource by spreading upon each time period the extra amount of non renewable resource while keeping the exhaustion length the same. For the same reasons when the renewable flow is increased, it is better for the society to increase the extraction rate of the non renewable resource, resulting in a shorter exhaustion length.